

SPECTRAL REPRESENTATIONS OF VERTEX TRANSITIVE GRAPHS, ARCHIMEDEAN SOLIDS AND FINITE COXETER GROUPS

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ABSTRACT. In this article, we study eigenvalue functions of varying transition probability matrices on finite, vertex transitive graphs. We prove that the eigenvalue function of an eigenvalue of fixed higher multiplicity has a critical point if and only if the corresponding spectral representation is equilateral. We also show how the geometric realisation of a finite Coxeter group as a reflection group can be used to obtain an explicit orthogonal system of eigenfunctions. Combining both results, we describe the behaviour of the spectral representations of the second highest eigenvalue function under the change of the transition probabilities in the case of Archimedean solids.

1. INTRODUCTION AND STATEMENT OF RESULTS

The main objects of interest in this paper are *spectral representations* associated to random walks on finite graphs (see Sections 1.1 and 1.2 for the definitions). We consider the particular case of vertex transitivity, which comprises the large class of Cayley graphs. In our main result (Theorem 1.3 below), we prove a correspondence between the critical points of an eigenvalue function (under the change of the invariant transition probabilities) and the points where the associated spectral representation is equilateral. In Sections 1.4 and 1.5, we specialise our considerations to finite Coxeter groups and one-skeleta of Archimedean solids.

1.1. Basic graph theoretical notation. Let $G = (V, E)$ be a finite, simple (i.e., no loops and multiple edges) graph with vertex set $V = \{1, \dots, n\}$ and set of undirected edges E . An edge is represented by a set $\{i, j\} \subset V$ with $i \neq j$. A (time reversible) random walk on G is given by a *symmetric* stochastic matrix $P = (p_{ij}) \in \mathbb{R}^{n \times n}$, where p_{ij} is the transition probability from vertex i to vertex j . For $i \neq j$, we require $p_{ij} = 0$ if $\{i, j\} \notin E$. Even though there are no loops in G , we allow the diagonal elements p_{ii} to be positive. (p_{ii} represents the probability for the random walk to stay at the vertex i .) The set of all matrices P of the above type are a convex subset of $\mathbb{R}^{n \times n}$, which we denote by Π_G . We think of a matrix $P \in \Pi_G$ as a linear operator on the vector space $l^2(G)$ of (real-valued) functions on the vertices,

i.e.,

$$Pf(i) = p_{ii}f(i) + \sum_{j \sim i} p_{ij}f(j),$$

where $j \sim i$ means that $\{i, j\} \in E$. The inner product on $l^2(G)$ is given by $\langle f, g \rangle = \sum_{i=1}^n f(i)g(i)$. Let $\sigma(P)$ denote the spectrum of P with eigenvalues

$$1 = \lambda_0(P) \geq \lambda_1(P) \geq \cdots \lambda_{n-1}(P) \geq -1,$$

counted with multiplicity. Let $f_0(i) = \frac{1}{\sqrt{n}}$. The Rayleigh quotient representation of the second highest eigenvalue function

$$(1) \quad \lambda_1(P) = \sup_{f \perp f_0, f \neq 0} \frac{\langle Pf, f \rangle}{\|f\|^2}$$

implies that $\lambda_1 : \Pi_G \rightarrow [-1, 1]$ is *convex* (see the proof of Proposition 1.2 in Section 2). The functions $\lambda_i : \Pi_G \rightarrow [-1, 1]$ are continuous (see, e.g., [14, Theorem (1,4)]), but these functions fail to be analytic at those points where eigenvalues of higher multiplicity bifurcate. We refer the reader to, e.g., [11, Chapter 2], for more information about these subtle regularity issues. The special operator $P = (p_{ij})$ with vanishing main diagonal ($p_{ii} = 0$ for all $i \in V$), and for which all other transition probabilities p_{ij} are equal to $1/\deg(i)$, is called the *canonical Laplacian*.

1.2. Spectral representations. The idea of a *spectral representation* is to use a higher multiplicity eigenvalue of the matrix P to obtain a "geometric realisation" of the combinatorial graph G in Euclidean space. Assume that $\lambda \in \sigma(P)$ is an eigenvalue of P of multiplicity k , and ϕ_1, \dots, ϕ_k is an orthonormal base of eigenfunctions of the eigenspace $\mathfrak{E}_\lambda(P)$. The corresponding spectral representation is the map

$$\Phi = \Phi_{P,\lambda} : V \rightarrow \mathbb{R}^k, \quad \Phi(i) = (\phi_1(i), \dots, \phi_k(i)),$$

i.e., the simultaneous evaluation of all eigenfunctions at a given vertex. The spectral representation depends on the choice of the orthonormal base only up to an orthonormal transformation in \mathbb{R}^k .

There are often striking geometric and spectral analogies between the discrete setting of graphs and the smooth setting of Riemannian manifolds. In the context of Riemannian manifolds, the simultaneous evaluation of eigenfunctions of the Laplacian were considered, for example, in the so-called *nice (minimal isometric) embeddings* of strongly harmonic manifolds into Euclidean spheres (see [4, Chapter 6G]).

Definition 1.1. A spectral representation $\Phi : V \rightarrow \mathbb{R}^k$ is faithful if Φ is injective. It is equilateral if all images of edges have the same Euclidean length, i.e.,

$$\|\Phi(i_1) - \Phi(j_1)\| = \|\Phi(i_2) - \Phi(j_2)\|,$$

for all pairs of edges $\{i_1, j_1\}, \{i_2, j_2\} \in E$, where $\|\cdot\|$ denotes the Euclidean norm.

A particularly strong faithfulness result for 3-connected, planar graphs in the case that the second highest eigenvalue has multiplicity three was obtained in [12].

1.3. Vertex transitive graphs. In this paper, we focus on finite *vertex transitive graphs*, i.e., we assume that the automorphism group $\text{Aut}(G)$ acts transitively on the vertex set V . Particular examples of vertex transitive graphs are Cayley graphs of groups. Below, we introduce equivalence classes of edges and, to have enough flexibility, we consider subgroups $\Gamma \subset \text{Aut}(G)$ which still act transitively on the vertices. We define a Γ -action on the space Π_G of matrices as follows:

$$(\gamma P)_{ij} = p_{\gamma i, \gamma j} \quad \text{for all } P = (p_{ij}) \in \Pi_G.$$

A random walk and its corresponding matrix $P \in \Pi_G$ is called Γ -invariant, if $\gamma P = P$ for all $\gamma \in \Gamma$. Note that the main diagonal (p_{11}, \dots, p_{nn}) of every Γ -invariant matrix P is constant. The large automorphism group of a vertex transitive graph makes the occurrence of eigenvalues of higher multiplicities for Γ -invariant matrices more likely, and it is natural to make use of connections between these eigenvalues and the representation theory of Γ .

The group Γ induces an equivalence relation on the set of edges: $\{i, j\} \in E$ is equivalent to all edges $\{\gamma i, \gamma j\}$ with $\gamma \in \Gamma$. The *multiplicity* of an equivalence class $[e] \subset E$ is the number of edges in $[e]$ meeting at the same vertex. Let $[e_1], \dots, [e_N]$ be the Γ equivalence classes of edges and m_1, \dots, m_N be its multiplicities. The set of Γ -invariant matrices in Π_G with vanishing main diagonal is a convex subset, which we identify with the simplex

$$(2) \quad \Delta_\Gamma = \{(x_1, \dots, x_N) \in [0, 1]^N \mid \sum_j m_j x_j = 1\}.$$

The point $X = (x_1, \dots, x_N) \in \Delta_\Gamma$ corresponds to the matrix $P_X = (p_{ij})$, given by

$$p_{ij} = \begin{cases} 0, & \text{if } i = j \text{ or } \{i, j\} \notin E, \\ x_k, & \text{if } \{i, j\} \in [e_k]. \end{cases}$$

For $P = (p_{ij}) \in \Pi_G$, let $G_P = (V, E_P)$ denote the subgraph of G with edges $E_P = \{\{i, j\} \in E \mid p_{ij} > 0\}$. Then, for every interior point $X \in \text{int}(\Delta_\Gamma)$, we have $G_{P_X} = G$ (since the entries p_{ij} associated to all edges $\{i, j\}$ are strictly positive), and the spectrum $\sigma(P_X)$ is symmetric with respect to the origin if and only if G_{P_X} is bipartite.

Let us now discuss the special case of Cayley graphs. A finite symmetric set $S \subset \Gamma$ of generators of a group Γ is called *minimal* if for every $s \in S$, $S - \{s, s^{-1}\}$ is no longer a set of generators. The Cayley graph of Γ with respect to S is denoted by $\text{Cay}(\Gamma, S)$, its vertices are the group elements, i.e., $V = \Gamma$, and two vertices γ, γ' are connected by an edge if and only if $\gamma' = \gamma s$ for some $s \in S$. If $S = \{s_1, \dots, s_r\}$ is a minimal symmetric set of

generators, we distinguish the generators of order 2 (since they appear only once in S) from the ones with higher order, by rewriting them as

$$(3) \quad S = \{s_1, \dots, s_\nu, \tau_1^{\pm 1}, \dots, \tau_\mu^{\pm 1}\},$$

with $\nu + 2\mu = r$. Note that the edges $\{e, \tau_j\}$ and $\{e, \tau_j^{-1}\}$ are equivalent, and the corresponding simplex Δ_Γ is given by

$$(4) \quad \Delta_\Gamma = \{(x_1, \dots, x_{\nu+\mu}) \in [0, 1]^N \mid \sum_{j=1}^{\nu} x_j + 2 \sum_{j=1}^{\mu} x_{\nu+j} = 1\}.$$

The following facts follow from the convexity of $\lambda_1 : \Pi_G \rightarrow [-1, 1]$ (see Section 2 for the proof).

Proposition 1.2. *Let $G = (V, E)$ be a finite, connected, simple graph and $\Gamma \subset \text{Aut}(G)$ be vertex transitive. Then a global minimum of $\lambda_1 : \Pi_G \rightarrow [-1, 1]$ is assumed at a matrix in Δ_Γ .*

If $G = \text{Cay}(\Gamma, S)$ is the Cayley graph of a finite group Γ with respect to a minimal symmetric set S of generators, then we have

$$(5) \quad \lim_{n \rightarrow \infty} \lambda_1(P_{X_n}) = 1$$

for every sequence $X_n \rightarrow \partial\Delta_\Gamma$, and a global minimum of λ_1 is assumed at an interior point of Δ_Γ .

Note that the above result does not rule out that λ_1 may also have other global minima at matrices $P \in \Pi_G - \Delta_\Gamma$.

Our main general result is the following relation between critical points of eigenvalue functions and equilateral spectral representations:

Theorem 1.3. *Let $G = (V, E)$ be a finite, connected, simple graph and $\Gamma \subset \text{Aut}(G)$ be vertex transitive. Let $U \subset \Delta_\Gamma$ be an open set and $\lambda : U \rightarrow [-1, 1]$ be a smooth function such that $\lambda(X) := \lambda(P_X)$ is an eigenvalue of P_X with fixed multiplicity $k \geq 2$ for all $X \in U$. $X_0 \in U$ is a critical point of the function λ if and only if the spectral representation $\Phi = \Phi_{P_{X_0}, \lambda(X_0)} : V \rightarrow S^{k-1}$ is equilateral.*

It is shown in Lemma 2.1 (see Section 2) that, for vertex transitive graphs, the image of every Γ -invariant spectral representation $\Phi = \Phi_{P_X, \lambda}$ (with $\lambda \in \sigma(P_X)$ of multiplicity k) lies on an Euclidean sphere $S^{k-1} \subset \mathbb{R}^k$, and that equivalent edges are mapped to segments with the same Euclidean length, i.e., $\|\Phi(\gamma i) - \Phi(\gamma j)\| = \|\Phi(i) - \Phi(j)\|$. The above theorem states that at critical points of the eigenvalue function all Euclidean images of edges have the same length (not only the equivalent ones).

Special examples of critical points are minima of a smooth function. As another example for similarities between graphs and Riemannian manifolds, we like to mention the following result in Riemannian geometry: The first non-zero Laplace-eigenvalue $\lambda_1(M) > 0$ of a closed Riemannian manifold (M, g) of dimension n with lower positive Ricci curvature is minimal if and

only if M is isometric to the n -dimensional round sphere (Obata's theorem, see [15] or [3]). Here we also have the phenomenon that a critical point of the eigenvalue function is assumed in the case of a very symmetric geometry.

Standard arguments in representation theory yield the following useful result:

Proposition 1.4. *Let Γ be a finite group with a minimal symmetric set of generators S given by (3) and $G = \text{Cay}(\Gamma, S)$ be the associated Cayley graph with the corresponding simplex Δ_Γ as in (4).*

Let $\rho : \Gamma \rightarrow O(k)$ be an irreducible representation, $\pi_r : \mathbb{R}^k \rightarrow \mathbb{R}$ be the projection to the r -th coordinate and $S^{k-1} \subset \mathbb{R}^k$ be the unit sphere. Let $p \in S^{k-1}$, $\lambda \in \mathbb{R}$, and $X = (x_1, \dots, x_{\nu+\mu}) \in \Delta_\Gamma$ such that

$$(6) \quad \lambda p = \sum_{j=1}^{\nu} x_j \rho(s_j) p + \sum_{j=1}^{\mu} x_{\nu+j} (\rho(\tau_j) p + \rho(\tau_j^{-1}) p).$$

Then the functions

$$\phi_r : \Gamma \rightarrow \mathbb{R}, \quad \phi_r(\gamma) := \pi_r(\rho(\gamma)p), \quad 1 \leq r \leq k$$

are pairwise orthogonal eigenfunctions of P_X for the eigenvalue λ satisfying $\|\phi_r\|^2 = \frac{|\Gamma|}{k}$.

Remarks 1.5. (a) *This result implies that if ϕ_1, \dots, ϕ_k span the whole eigenspace $\mathfrak{E}_\lambda(P_X)$, then the associated spectral representation $\Phi : \Gamma \rightarrow S^{k-1}$ coincides with the orbit map $\Phi(\gamma) = \rho(\gamma)p_0$ of the rescaled point $p_0 = \sqrt{\frac{k}{|\Gamma|}} p \in \mathbb{R}^r$.*

(b) *It can be shown, in the weaker case of a non-orthogonal irreducible representation $\rho : \Gamma \rightarrow GL(k, \mathbb{R})$, that the functions ϕ_r are still a family of linear independent eigenfunctions of P_X .*

1.4. Finite Coxeter groups. Let us now consider the special case of a finite irreducible Coxeter group $\Gamma = \langle S = \{s_1, \dots, s_k\} \mid (s_i s_j)^{m_{ij}} = e \rangle$ of rank k with $m_{ij} = m_{ji} \geq 2$ and $m_{ii} = 2$, i.e., $s_i^2 = e$. It was suggested in [13, Problem 10.8.7] to study the eigenvalues (or at least λ_1) of the canonical Laplacian for Coxeter groups, and results related to this problem can be found in the papers [2], [1], and [10]. (Another important spectral invariant is the Kazhdan constant. See [10] for the Kazhdan constants of all finite Coxeter groups. For infinite Coxeter groups, it was proved in [5] that they do not have Kazhdan property (T).)

Let $\Gamma \hookrightarrow O(k)$ be the geometric realisation of Γ as finite reflection group. The associated Cayley graph $G = \text{Cay}(\Gamma, S)$ is bipartite, since all relations of a Coxeter group have even length. Let $\sigma_j \in O(k)$ be the reflections corresponding to the generators s_j and n_1, \dots, n_k be the associated simple roots. Let

$$(7) \quad p_j = (-1)^{j-1} n_1 \times \dots \times \widehat{n_j} \times \dots \times n_k,$$

where $v_1 \times \cdots \times v_{k-1}$ denotes the $(k-1)$ -ary analogue of the cross product in \mathbb{R}^k , and the hat over n_j in (7) means that this term is dropped. Then the open cone

$$\mathcal{F} := \{\alpha_1 p_1 + \cdots + \alpha_k p_k \mid \alpha_1, \dots, \alpha_k > 0\} \subset \mathbb{R}^k,$$

is a fundamental domain of the Γ -action on \mathbb{R}^k . Γ preserves the unit sphere S^{k-1} , and a spherical fundamental domain is given by $\mathcal{F}_0 = \mathcal{F} \cap S^{k-1}$. Let $V = \det(n_1, \dots, n_k)$. Without loss of generality, we can assume that $V > 0$, for otherwise we simply permute the set of generators. The following result is a consequence of Proposition 1.4.

Corollary 1.6. *Let Γ be a finite, irreducible Coxeter group and $\mathcal{F}_0 \subset S^{k-1}$ and $\Delta = \Delta_\Gamma$ be as above. Then there exists smooth maps $\Psi_\Delta : \mathcal{F}_0 \rightarrow \text{int}(\Delta)$ and $\Psi_\lambda : \mathcal{F}_0 \rightarrow (0, 1)$, with Ψ_Δ bijective, such that, for every $p = \sum \alpha_j p_j \in \mathcal{F}_0$, the functions $\phi_i(\gamma) := \pi_i(\gamma p)$ are pairwise orthogonal eigenfunctions of P_X on $\text{Cay}(\Gamma, S)$ for the eigenvalue $\lambda = \Psi_\lambda(p)$, where $X = \Psi_\Delta(p)$. Moreover, $\|\phi_r\|^2 = \frac{|\Gamma|}{k}$ and the composition $\Psi_\lambda \circ \Psi_\Delta^{-1} : \text{int}(\Delta) \rightarrow (0, 1)$ is analytic. The simultaneous evaluation*

$$\Phi(\gamma) : \Gamma \rightarrow S^{k-1}, \quad \Phi(\gamma) = (\phi_1(\gamma), \dots, \phi_k(\gamma)) = \gamma p$$

is faithful, and the Euclidean lengths of the images of equivalence classes of edges under Φ are given by

$$\|p - \sigma_j(p)\| = 2\alpha_j V.$$

Remark 1.7. *The explicit description of the maps Ψ_Δ and Ψ_λ is given by the equations (16), (17), and (18) in Section 3.*

Proposition 1.8. *Let Γ be one of the Coxeter groups A_3 , B_3 or H_3 . Then the map $\Psi_\lambda \circ \Psi_\Delta^{-1} : \text{int}(\Delta) \rightarrow (0, 1)$ from Corollary 1.6 coincides with the second highest eigenvalue function λ_1 , and $\lambda_1(X)$ has multiplicity three for all $X \in \text{int}(\Delta)$.*

Note that the value of $\lambda_1(X)$ has a well known *dynamical interpretation*: Our Cayley graphs are bipartite, i.e., we have a partition $V = V_0 \cup V_1$. $\lambda_1(X)$ measures the convergence rate of the corresponding random walk to the equidistribution (*mixing rate*) on each set of vertices V_i under even time steps (even time steps are needed because of the bipartiteness).

The proof of Proposition 1.8 is based on elegant arguments of van der Holst [17]. He used these arguments to give a direct combinatorial proof of Colin de Verdière's planarity characterisation " $\mu(G) \leq 3$ ".

Remark 1.9. *We conjecture that Proposition 1.8 generalises to all finite Coxeter groups Γ , with the multiplicity of $\lambda_1(X)$ being equal to the rank of Γ . The validity of this conjecture together with our main result (Theorem 1.3) would allow us to explicitly determine, for all finite Coxeter groups, the transition probabilities of a random walk with the fastest mixing rate on the corresponding Cayley graphs. In fact, this is precisely how we will prove Theorem 1.10 below.*

1.5. Archimedean solids. The Cayley graph of the Coxeter groups A_3 , B_3 and H_3 (with respect to their set of standard generators $\{s_1, s_2, s_3\}$) coincide, combinatorially, with the one-skeleta of the Archimedean solids with the vertex configurations $(4, 6, 6)$, $(4, 6, 8)$, and $(4, 6, 10)$, respectively.

Archimedean solids are polyhedra in \mathbb{R}^3 such that all faces are regular polygons, and which have a symmetry group acting transitively on the vertices. (Note, however, that the prisms, antiprisms and Platonic solids, which also have these properties, are excluded). The 13 Archimedean solids are classified via their *vertex configurations*: The vertex configuration (m, n, k) stands for the solid where an m -gon, an n -gon and a k -gon (in this order) meet at every vertex. We will use this notation also for Platonic solids (e.g., the icosahedron is denoted by $(3, 3, 3, 3, 3)$). The spectra of the canonical Laplacians (on the one-skeleta) of all Archimedean solids were explicitly calculated in [16]. For all these graphs, the second highest eigenvalue of the canonical Laplacian has multiplicity three. The corresponding spectral representation is faithful and represents a polyhedron in \mathbb{R}^3 (this follows, e.g., from the general result in [12]), but this polyhedron is generally *not equilateral*. It is natural to study the deformation of this polyhedron under changes of the Γ -invariant transition probabilities (assuming that the multiplicity of λ_1 does not change), and to find points at which the spectral representation is equilateral.

We will carry this out in the case of the largest Archimedean solid, namely the truncated icosidodecahedron $(4, 6, 10)$. We will also explain, how the corresponding results read in the case of the Archimedean solids $(4, 6, 8)$ and $(4, 6, 6)$. The proofs for these cases are completely analogous.

Let $G = (V, E)$ be the one-skeleton of the Archimedean solid $(4, 6, 10)$. The automorphism group of G is the full icosahedral group and acts simply transitively on the vertex set V , and is isomorphic to H_3 . Considering G as a planar graph, its faces are 4-, 6- and 10-gons. G is 3-connected, has 120 vertices and every vertex has degree three (see Figure 1 below). Let $\varphi = \frac{1+\sqrt{5}}{2} = 2 \cos \frac{\pi}{5}$ be the golden ratio. Our previous results imply the following facts for λ_1 .

Theorem 1.10. *Let G be the 1-skeleton of the Archimedean solid $(4, 6, 10)$ and $\Gamma = \text{Aut}(G)$. The simplex of Γ -invariant transition probabilities is*

$$\Delta = \Delta_\Gamma = \{(x, y, z) \mid x, y, z \geq 0, x + y + z = 1\},$$

where x, y, z are the transition probabilities for the edge-equivalence classes separating 4- and 6-gons, 4- and 10-gons, and 6- and 10-gons, respectively. Then the restriction of $\lambda_1 : \Pi_G \rightarrow [-1, 1]$ to $\text{int}(\Delta) \subset \Pi_G$ is analytic and strictly convex, and $\lambda_1(X)$ has multiplicity three for all $X \in \text{int}(\Delta)$. Moreover, $X_0 = \frac{1}{14+5\varphi}(5, 3+3\varphi, 6+2\varphi)$ is the unique point in Δ at which λ_1 assumes its global minimum with

$$\lambda_1(X_0) = \frac{10 + 7\varphi}{14 + 5\varphi}.$$

The corresponding spectral representation $\Phi_{X_0} : V \rightarrow S^2$ is faithful and equilateral.

Let us stress, again, that for $\hat{X} = (1/3, 1/3, 1/3) \in \Delta$, the above Theorem implies that the spectral representation of $P_{\hat{X}}$ for $\lambda_1(\hat{X})$ does *not* reproduce the Archimedean solid, one has to choose the point $X_0 \in \Delta$ instead (see Figure 1).

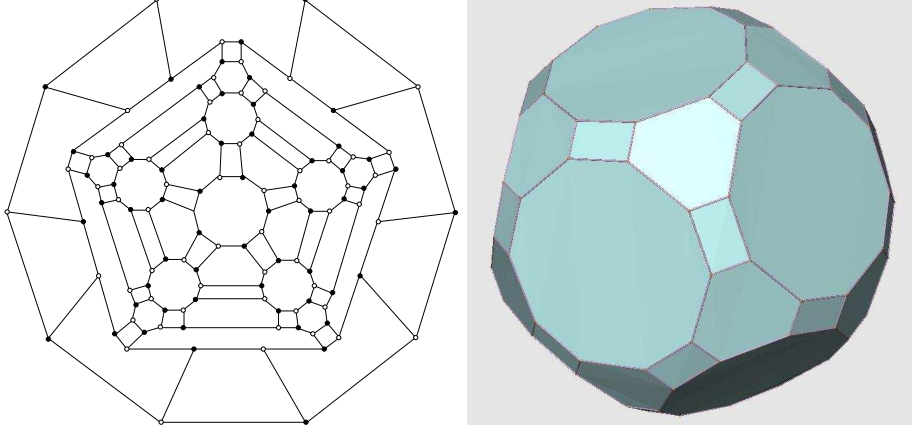


FIGURE 1. The 1-skeleton of the Archimedean solid (4, 6, 10) and the (non-equilateral) spectral representation of the canonical Laplacian for the second highest eigenvalue.

Remark 1.11. *There are analogous versions of Theorem 1.10 in the cases (4, 6, 8) and (4, 6, 6). The full symmetry group of both solids (4, 6, 8) and (4, 6, 6) is the full octahedral group, but it is better to view (4, 6, 6) as a polyhedron with the full tetrahedral group (which is a subgroup of the full octahedral group) as its symmetry group, by distinguishing its hexagonal faces with the help of two colours (say, yellow and blue), such that adjacent 6-gons have different colours. In this case the solid (4, 6, 6) is also called the omnitruncated tetrahedron and has three equivalence classes of edges (separating 4-gons and yellow 6-gons, 4-gons and blue 6-gons, yellow and blue 6-gons), just as the solid (4, 6, 8) and (4, 6, 10). The corresponding explicit values for X_0 and $\lambda_1(X_0)$ are*

$$X_0 = \frac{1}{13 + 6\sqrt{2}}(4 + \sqrt{2}, 3 + 3\sqrt{2}, 6 + 2\sqrt{2}) \quad \text{and} \quad \lambda_1(X_0) = \frac{11 + 6\sqrt{2}}{13 + 6\sqrt{2}}$$

in the case (4, 6, 8) and

$$X_0 = \left(\frac{3}{10}, \frac{3}{10}, \frac{2}{5}\right) \quad \text{and} \quad \lambda_1(X_0) = \frac{4}{5}$$

in the case (4, 6, 6).

Finally, we describe the behaviour of spectral representations $\Phi_X : V \rightarrow S^2$, as $X \in \Delta$ moves towards the boundary $\partial\Delta$.

Theorem 1.12. *Let G, Δ, X_0 be as in Theorem 1.10. Then there are three explicitly given curves $C_1, C_2, C_3 \subset \Delta$, which meet in X_0 and have the following property: For every $X \in C_i$, the lengths of two of the three equivalence classes of Euclidean edges in the spectral representation of P_X for the eigenvalue $\lambda_1(X)$ coincide.*

As X_n converges to the corresponding vertex of the simplex Δ along the curve C_i , the spectral representations converge to equilateral realisations of the Archimedean solids $(3, 10, 10)$, $(5, 6, 6)$ and $(3, 4, 5, 4)$, respectively.

For any sequence $X_n \in \text{int}(\Delta)$ converging to an interior point of the boundary edge of the simplex Δ , the spectral representations converge to the equilateral realisations of one of the solids $(3, 3, 3, 3, 3)$, $(5, 5, 5)$ and $(3, 5, 3, 5)$.

These convergence properties are illustrated in Figure 2.

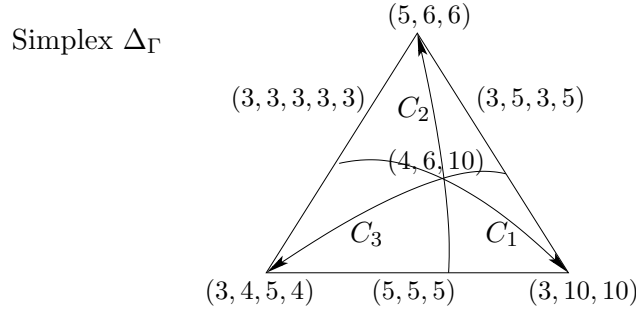


FIGURE 2. Convergence behaviour of Φ_X as $X \rightarrow \partial\Delta$.

Figure 3 shows the spectral representations of P_X for three points X along the curve C_2 , illustrating the transition from the dodecahedron $(5, 5, 5)$ to the buckyball $(5, 6, 6)$.

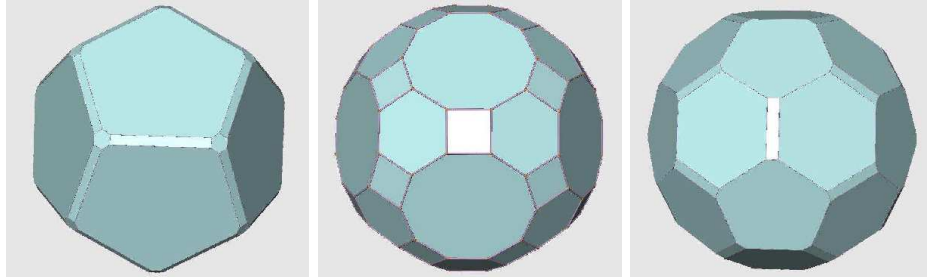


FIGURE 3. Spectral representations of P_X (for points X along C_2) for the second highest eigenvalue.

Remark 1.13. *The analogous versions of Theorem 1.12 for the Archimedean solids $(4, 6, 8)$ and $(4, 6, 6)$ are illustrated in Figure 4 below. The common symmetry group of all solids in the diagram containing $(4, 6, 8)$ is the full octahedral group. In the diagram containing $(4, 6, 6)$, we need to colour the hexagons in the solid $(4, 6, 6)$ with two different colours (as described in Remark 1.11) and, similarly, we have to colour the triangles of the solid $(3, 4, 3, 4)$ with two colours such that triangles meeting in a vertex have different colours (and refer to $(3, 4, 3, 4)$ then as the cantellated tetrahedron), so that the common symmetry group of all solids in this diagram is the full tetrahedral group.*

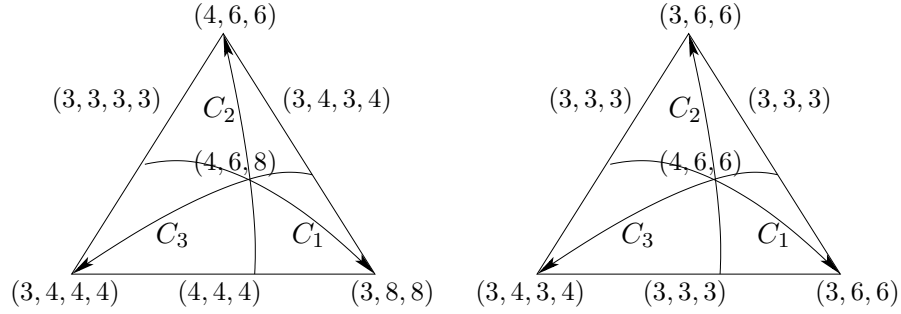


FIGURE 4. Boundary convergence behaviour in the cases $(4, 6, 8)$ and $(4, 6, 6)$.

1.6. Structure of the article. Section 2 provides the proofs of Propositions 1.2 and 1.4 and of our Main Theorem 1.3. In Sections 3 and 4, we prove Corollary 1.6 and Proposition 1.8. Finally, Section 5 presents the proofs of Theorems 1.10 and 1.12.

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2. PROOFS OF THE GENERAL RESULTS

Let us start with a convexity proof of $\lambda_1 : \Pi_G \rightarrow [-1, 1]$, which is used to show that λ_1 assumes a global minimum in Δ_Γ .

Proof of Proposition 1.2: First note that, for $P, P' \in \Pi_G$ and $\alpha \in [0, 1]$,

$$\frac{\langle (\alpha P + (1 - \alpha)P')f, f \rangle}{\|f\|^2} = \alpha \frac{\langle Pf, f \rangle}{\|f\|^2} + (1 - \alpha) \frac{\langle P'f, f \rangle}{\|f\|^2},$$

which implies the convexity of $\lambda_1 : \Pi_G \rightarrow [-1, 1]$ by taking supremums on both sides and using the characterisation (1).

Note also that Π_G is compact, and the continuous function $\lambda_1 : \Pi_G \rightarrow [-1, 1]$ must have a global minimum at some point $P \in \Pi_G$. If $P \notin \Delta_\Gamma$, then consider the Γ -invariant matrix $P' = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma P \in \Pi_G$, and we conclude $\lambda_1(P') \leq \lambda_1(P)$ by the convexity of λ_1 . P' may have a non-vanishing

(constant) main diagonal. Nevertheless, we can write $P' = \beta \text{Id} + (1 - \beta)P''$ with appropriate $P'' \in \Delta_\Gamma$ and $\beta \geq 0$. This implies that

$$\lambda_1(P') - \lambda_1(P'') = \beta(1 - \lambda_1(P'')) \geq 0,$$

which shows that λ_1 assumes also a global minimum at $P'' \in \Delta_\Gamma$.

Now, assume that $G = \text{Cay}(\Gamma, S)$ and that S is a minimal set of generators. The minimality of S implies that, for every $X \in \partial\Delta_\Gamma$, the graph G_{P_X} consists of more than one connected component and, therefore, $\lambda_0(P_X) = \lambda_1(P_X) = 1$. Then (5) follows from the continuity of the function λ_1 . For every interior point $X \in \Delta_\Gamma$ we have $\lambda_1(P_X) < 1$, since G_{P_X} is connected.

□

Our next goal is the proof that Γ -invariant spectral representations map all vertices onto a sphere and that equivalent edges are mapped to Euclidean segments of the same length.

Lemma 2.1. *Let $G = (V, E)$ be a simple finite graph and $\Gamma \subset \text{Aut}(G)$ be vertex transitive with N equivalence classes of edges, and Δ_Γ be as in (2). Let $X \in \Delta_\Gamma$ and $\lambda \in [-1, 1]$ be an eigenvalue of multiplicity $k \geq 2$ of the operator P_X . Let $\Phi = \Phi_{X, \lambda} : V \rightarrow \mathbb{R}^k$ be the associated spectral representation. Then there exist constants $c > 0$, $c_1, \dots, c_N \geq 0$ such that*

- (a) *for all vertices $i \in V$: $\|\Phi(i)\| = c$,*
- (b) *for all edges $\{i, j\} \in E$ in the l -th equivalence class*

$$\|\Phi(i) - \Phi(j)\| = c_l.$$

Proof: Let ϕ_1, \dots, ϕ_k be the orthonormal basis of eigenfunctions defining $\Phi = (\phi_1, \dots, \phi_k)^\top$ (we consider $\Phi(i)$ as a column vector). Let $\gamma \in \Gamma$ be fixed and $\psi_r = \phi_r \circ \gamma : V \rightarrow \mathbb{R}$. One easily checks that ψ_1, \dots, ψ_k are also an orthonormal basis satisfying $P_X \psi_r = \lambda \psi_r$. Consequently, there exists a matrix $C = (c_{rs}) \in O(k)$ such that $\psi_r = \sum_{s=1}^k c_{rs} \phi_s$. This implies $\Phi(\gamma i) = C\Phi(i)$ and

$$(8) \quad \langle \Phi(\gamma i), \Phi(\gamma j) \rangle = \langle C\Phi(i), C\Phi(j) \rangle = \langle \Phi(i), \Phi(j) \rangle.$$

(8) implies (a) by choosing $i = j$ and using the vertex transitivity of Γ . (b) follows from (a), (8) and

$$\|\Phi(i) - \Phi(j)\|^2 = \|\Phi(i)\|^2 - 2\langle \Phi(i), \Phi(j) \rangle + \|\Phi(j)\|^2.$$

□

Before entering into the proof of our main result, let us remark that the above identity (8) can be rewritten as

$$(9) \quad \sum_{r=1}^k \phi_r(i) \phi_r(j) = \sum_{r=1}^k \phi_r(\gamma i) \phi_r(\gamma j) \quad \text{for all } i, j \in V \text{ and } \gamma \in \Gamma.$$

Moreover, observe that the following identity is an immediate consequence of the vertex transitivity of Γ , the left coset decomposition $\Gamma = \gamma_1 \Gamma_1 \cup$

$\gamma_2\Gamma_1 \cup \dots \cup \gamma_n\Gamma_1$, where $\Gamma_1 \subset \Gamma$ is the stabilizer of $1 \in V$, and the relation $|\Gamma| = n|\Gamma_1|$:

$$(10) \quad \sum_{i=1}^n f(i) = \frac{n}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma 1).$$

Proof of Theorem 1.3: For simplicity, we discuss the key arguments for the special choice of the first and second equivalence class of edges. The proof for two arbitrary equivalence classes is completely analogous.

Let $\{1, i_1\}, \dots, \{1, i_p\}$ be all edges adjacent to $1 \in V$ in the first equivalence class of edges (note that $p = m_1$). Let $\{1, j_1\}, \dots, \{1, j_q\}$ be all edges adjacent to $1 \in V$ in the second equivalence class of edges (note that $q = m_2$). We conclude from (9) that

$$(11) \quad \sum_{r=1}^k \phi_r(1)\phi_r(i_1) = \sum_{r=1}^k \phi_r(1)\phi_r(i_2) = \dots = \sum_{r=1}^k \phi_r(1)\phi_r(i_p),$$

and the same identity holds for the edges in the second equivalence class.

Let $X_0 \in U$ an arbitrary point (not necessarily a critical point of λ_1), $\xi = (\frac{1}{p}, -\frac{1}{q}, 0, \dots, 0) \in \mathbb{R}^N$ and $X_t := X_0 + t\xi \in U$ for $t \in (-\epsilon, \epsilon)$ and $\epsilon > 0$ suitably small. For simplicity of notation, we introduce $P(t) := P_{X_t}$, $\lambda(t) = \lambda(X_t)$. Let $\phi_{1,t}, \dots, \phi_{k,t}$ be an orthonormal basis of the eigenspace $\mathfrak{E}_{\lambda(t)}(P(t))$, depending smoothly on t . Let $\phi_r = \phi_{r,0}$. Note that $P'(0) = (c_{ij})$ with

$$c_{\gamma 1, \gamma i} = \begin{cases} \frac{1}{p}, & \text{if } i \in \{i_1, \dots, i_p\}, \\ -\frac{1}{q}, & \text{if } i \in \{j_1, \dots, j_q\}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $\gamma \in \Gamma$ and $i \in V$.

Let $r \in \{1, \dots, k\}$. By the orthonormality of the functions ϕ_r , we have $\langle \phi_r, \frac{\partial}{\partial t} \big|_{t=0} \phi_{r,t} \rangle = 0$. Using this and the symmetry of $P(t)$, we obtain, by differentiating $\lambda(t) = \langle P(t)\phi_{r,t}, \phi_{r,t} \rangle$ at $t = 0$:

$$\begin{aligned} \lambda'(0) = \langle P'(0)\phi_r, \phi_r \rangle &= \sum_{i=1}^n \phi_r(i) \left(\sum_{j=1}^n c_{ij} \phi_r(j) \right) \\ &= \frac{n}{|\Gamma|} \sum_{\gamma \in \Gamma} \phi_r(\gamma 1) \left(\sum_{j=1}^n c_{\gamma 1, \gamma j} \phi_r(\gamma j) \right) \quad \text{using (10)} \\ &= \frac{n}{|\Gamma|} \sum_{\gamma \in \Gamma} \phi_r(\gamma 1) \left(\frac{1}{p} \sum_{s=1}^p \phi_r(\gamma i_s) - \frac{1}{q} \sum_{s=1}^q \phi_r(\gamma j_s) \right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle \Phi(1), \Phi(i_1) \rangle &= \frac{1}{p} \sum_{s=1}^p \sum_{r=1}^k \phi_r(1) \phi_r(i_s) \quad \text{using (11)} \\ &= \sum_{r=1}^k \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \phi_r(\gamma 1) \frac{1}{p} \sum_{s=1}^p \phi_r(\gamma i_s) \quad \text{using (9)}.\end{aligned}$$

Combining both results, we obtain

$$\langle \Phi(1), \Phi(i_1) \rangle - \langle \Phi(1), \Phi(j_1) \rangle = \frac{k}{n} \lambda'(0).$$

This implies that we have $\|\Phi(1) - \Phi(i_1)\| = \|\Phi(1) - \Phi(j_1)\|$ if and only if $\lambda'(0) = 0$.

Since the above arguments hold for any choice of equivalence classes of edges, we have $c_1 = \dots = c_N$ in Lemma 2.1 above (i.e., an equilateral spectral representation) if and only if the derivative of λ at X_0 vanishes in all directions of the simplex, i.e., if $X_0 \in U$ is a critical point of λ . \square

The proof of Proposition 1.4 is based on the following lemma:

Lemma 2.2. *Let Γ be a finite group, $\rho : \Gamma \rightarrow O(k)$ be an irreducible representation and, as before, $\langle \cdot, \cdot \rangle$ be the standard inner product in \mathbb{R}^k . For any non-zero vector $p \in \mathbb{R}^k$ there is a constant $C_p > 0$ such that*

$$\sum_{\gamma \in \Gamma} \langle \rho(\gamma)p, v \rangle \langle \rho(\gamma)p, w \rangle = C_p \langle v, w \rangle$$

for all $v, w \in \mathbb{R}^k$.

Proof: The expression

$$\langle v, w \rangle_p := \sum_{\gamma \in \Gamma} \langle \rho(\gamma)p, v \rangle \langle \rho(\gamma)p, w \rangle$$

is obviously a symmetric bilinear form. The form is positive definite, since $\langle v, v \rangle_p = 0$ implies that v is perpendicular (w.r.t. the standard inner product) to $\text{span}\{\rho(\gamma)p \mid \gamma \in \Gamma\}$. Irreducibility of ρ implies that $\text{span}\{\rho(\gamma)p \mid \gamma \in \Gamma\} = \mathbb{R}^k$, so $v = 0$. Therefore, there exists a positive definite symmetric matrix A such that

$$\langle v, w \rangle_p = \langle Av, w \rangle.$$

Let $\gamma_0 \in \Gamma$. Then

$$\langle \rho(\gamma_0)v, \rho(\gamma_0)w \rangle_p = \sum_{\gamma \in \Gamma} \langle \rho(\gamma_0^{-1}\gamma)p, v \rangle \langle \rho(\gamma_0^{-1}\gamma)p, w \rangle = \langle v, w \rangle_p,$$

i.e., $\langle \cdot, \cdot \rangle_p$ is $\rho(\Gamma)$ -invariant, and we have $A\rho(\gamma) = \rho(\gamma)A$ for all $\gamma \in \Gamma$. Since ρ is irreducible, we conclude from Schur's lemma that A is of the form $C_p \cdot \text{Id}$ with a constant $C_p > 0$. This finishes the proof of the lemma. \square

Proof of Proposition 1.4: Note that the vertices of $G = \text{Cay}(\Gamma, S)$ are the group elements, and that

$$P_X f(\gamma) = \sum_{j=1}^{\nu} x_j f(\gamma s_j) + \sum_{j=1}^{\mu} x_{\nu+j} (f(\gamma \sigma_j) + f(\gamma \sigma_j^{-1})).$$

This implies that

$$\begin{aligned} P_X \phi_r(\gamma) &= \sum_{j=1}^{\nu} x_j \pi_r(\rho(\gamma s_j)p) + \sum_{j=1}^{\mu} x_{\nu+j} (\pi_r(\rho(\gamma \sigma_j)p) + \pi_r(\rho(\gamma \sigma_j^{-1})p)) \\ &= \pi_r \left(\rho(\gamma) \left(\sum_{j=1}^{\nu} x_j \rho(s_j)p + \sum_{j=1}^{\mu} x_{\nu+j} (\rho(\sigma_j)p + \rho(\sigma_j^{-1})p) \right) \right) \\ &= \lambda \pi_r(\rho(\gamma)p) = \lambda \phi_r(\gamma), \end{aligned}$$

by using (6). This shows that ϕ_r is an eigenfunction of P_X for the eigenvalue λ . The orthogonality of the functions ϕ_r is a straightforward application of Lemma 2.2:

$$\langle \phi_r, \phi_s \rangle = \sum_{\gamma \in \Gamma} \pi_r(\rho(\gamma)p) \pi_s(\rho(\gamma)p) = \sum_{\gamma \in \Gamma} \langle \rho(\gamma)p, e_r \rangle \langle \rho(\gamma)p, e_s \rangle = C_P \langle e_r, e_s \rangle,$$

where e_1, \dots, e_k denotes the standard basis in \mathbb{R}^k . Now let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$. Let A be the $(k \times n)$ matrix whose columns are the vectors $\rho(\gamma_j)p \in S^{k-1}$. Then the rows of A represent the functions ϕ_r , and we have

$$\sum_{r=1}^k \|\phi_r\|^2 = \sum_{j=1}^n \|\rho(\gamma_j)p\|^2 = n = |\Gamma|.$$

This shows that $\|\phi_r\|^2 = \frac{|\Gamma|}{k}$. □

Remark 2.3. Assume that ρ in Proposition 1.4 is irreducible but not orthogonal, i.e., $\rho : \Gamma \rightarrow GL(k, \mathbb{R})$. The above proof still shows that the functions ϕ_r are eigenfunctions. Let A be the $(k \times n)$ matrix as in the proof. Then the irreducibility of ρ implies that the columns span all of \mathbb{R}^k , i.e., the rank of A is k . But this means that the functions ϕ_r (the k rows of A) must be linearly independent.

3. PROOF OF COROLLARY 1.6

We start with a finite, irreducible Coxeter group. This implies that the geometric realisation $\Gamma \hookrightarrow O(k)$ is an irreducible, faithful representation. Note that we have $\langle n_i, n_j \rangle = -\cos \frac{\pi}{m_{ij}}$, where m_{ij} is the order of the element $s_i s_j$. Since Γ is a finite Coxeter group, $M = (\langle n_i, n_j \rangle)$ is a positive definite, symmetric matrix. Writing $M = \text{Id} - C$ with a symmetric matrix $C < \text{Id}$ (as quadratic forms) whose entries are all non-negative, we obtain $M^{-1} = \sum_{s=0}^{\infty} C^s$. Irreducibility implies that for every position $1 \leq i, j \leq k$, there

is an $s \geq 0$ such that $(C^s)_{ij} > 0$. This implies that all entries of M^{-1} are strictly positive. Recall that $V := \det(n_1, \dots, n_k) > 0$. We define, as in (7),

$$p_j = (-1)^{j-1} n_1 \times \dots \times \widehat{n_j} \times \dots \times n_k.$$

The vectors p_j may all have different Euclidean lengths. We have, by construction $\langle n_i, p_j \rangle = V \delta_{ij}$. Let

$$\Delta = \Delta_\Gamma = \{(x_1, \dots, x_k) \mid x_j \geq 0, \sum_j x_j = 1\}$$

be the simplex associated to the Cayley graph $\text{Cay}(\Gamma, S)$.

Our aim is to construct the maps $\Psi_\Delta : \mathcal{F}_0 \rightarrow \text{int}(\Delta)$ and $\Psi_\lambda : \mathcal{F}_0 \rightarrow (0, 1)$: Any point $p \in \mathcal{F}_0$ can be expressed uniquely as

$$p = \alpha_1 p_1 + \dots + \alpha_k p_k,$$

with $\alpha_1, \dots, \alpha_k > 0$. We will show that there is a unique choice of $X = (x_1, \dots, x_k) \in \text{int}(\Delta)$ and $\lambda \in (0, 1)$ such that

$$(12) \quad \lambda p = \sum_j x_j \sigma_j(p).$$

We then define $\Psi_\Delta(p) = X$ and $\Psi_\lambda(p) = \lambda$. The construction will show that X and λ depend smoothly on the coordinates α_j . Applying Proposition 1.4 yields the results stated in the Corollary. It then only remains to prove that Ψ_Δ is bijective and that the composition $\Psi_\lambda \circ \Psi_\Delta^{-1}$ is analytic.

Since $\sigma_j(p) = p - 2\langle p, n_j \rangle n_j = p - 2\alpha_j V n_j$, we immediately see that $\|p - \sigma_j(p)\| = 2\alpha_j V$. Moreover, (12) translates into

$$(13) \quad \lambda p = (x_1 + \dots + x_k)p - 2V \sum_j \alpha_j x_j n_j.$$

This means that we need to find a unique $(x_1, \dots, x_k) \in \Delta$ and $\mu \in \mathbb{R}$ such that

$$(14) \quad \sum_j \alpha_j x_j n_j = \mu \sum_j \alpha_j p_j,$$

and then set $\lambda = x_1 + \dots + x_k - 2V\mu$. Taking inner products with the simple roots n_1, \dots, n_k , and bringing everything in a matrix equation, we end up with the equivalent equation

$$(15) \quad M \begin{pmatrix} \alpha_1 x_1 \\ \vdots \\ \alpha_k x_k \end{pmatrix} = \mu V \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}.$$

Obviously, this equation is homogeneous, i.e., if (x_1, \dots, x_k, μ) is a solution then so is $(cx_1, \dots, cx_k, c\mu)$ for any constant c . We first seek for the unique solution (x'_1, \dots, x'_k) of (15) for the choice $\mu = 1$. (x'_1, \dots, x'_k) will not be a point in Δ , and we obtain the correct solution by way of rescaling. Using

the fact that $M^{-1} = \text{Id} + D$, where all diagonal entries of D are non-negative and all off-diagonal are strictly positive, we end up with the inequality

$$(16) \quad \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \end{pmatrix} = V \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_k \end{pmatrix}^{-1} M^{-1} \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_k \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} > V \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

This shows that any choice $\alpha_1, \dots, \alpha_k > 0$ leads to a strictly positive vector (x'_1, \dots, x'_k) , and that

$$(17) \quad \lambda' := x'_1 + \dots + x'_k - 2V > 0.$$

For $p = \sum_j \alpha_j p_j \in \mathcal{F}_0$, we first calculate $x'_1, \dots, x'_k, \lambda' > 0$ via the equations (16) and (17), and then apply the rescaling to obtain

$$(18) \quad \Psi_\Delta(p) = \frac{1}{\sum_j x'_j} (x'_1, \dots, x'_k) \in \text{int}(\Delta), \quad \Psi_\lambda(p) = \frac{\lambda'}{\sum_j x'_j} \in (0, 1).$$

Next we show that $\Psi_\Delta : \mathcal{F}_0 \rightarrow \text{int}(\Delta)$ is bijective. Choose $X = (x_1, \dots, x_k) \in \text{int}\Delta$. An equivalent reformulation of (15) is

$$(19) \quad \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} = \mu V \begin{pmatrix} x_1^{-1} & & \\ & \ddots & \\ & & x_k^{-1} \end{pmatrix} M^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}.$$

Note that $V \text{diag}(x_1^{-1}, \dots, x_k^{-1}) M^{-1}$ is a matrix with all its entries strictly positive. Therefore, we can apply Perron-Frobenius theory and conclude that there is a unique Perron-Frobenius eigenvector $(\alpha_1, \dots, \alpha_k)$, scaled in such a way that $p = \sum_j \alpha_j p_j \in S^{k-1}$. Since $\alpha_j > 0$, we conclude that $p \in \mathcal{F}_0$. This shows that every $X \in \text{int}(\Delta)$ has a unique preimage under Ψ_Δ .

Finally, note that the composition $\Psi_\lambda \circ \Psi_\Delta^{-1} : \text{int}(\Delta) \rightarrow (0, 1)$ is given by $(\sum_j x_j) - 2V\mu = 1 - 2V\mu$, where μ is the Perron-Frobenius eigenvalue of the matrix $V \text{diag}(x_1^{-1}, \dots, x_k^{-1}) M^{-1}$. Since this eigenvalue has always multiplicity one, it depends analytically on the weights x_1, \dots, x_k , by the analytic version of the Implicit Function Theorem. This finishes the proof of Corollary 1.6. \square

4. PROOF OF PROPOSITION 1.8

Let us indicate the strategy of proof: Assume that we have $\Psi_\lambda \circ \Psi_\Delta^{-1}(\widehat{X}) = \lambda_1(\widehat{X})$ for a certain point $\widehat{X} \in \text{int}(\Delta)$, and that $\lambda_1 : \text{int}(\Delta) \rightarrow [0, 1]$ would always have multiplicity one. Under these assumptions, λ_1 could not touch or have a cross-over with the eigenvalue function λ_2 , since at such a point the multiplicity of λ_1 would be ≥ 2 . Therefore, the continuity of λ_1 and $\Psi_\lambda \circ \Psi_\Delta^{-1}$ would necessarily imply that both functions agree. Note, however, that the proof is more complicated, since λ_1 is not simple.

Our first goal is to prove Corollary 4.2 below. We follow the notation used in [17], but recall them for the reader's convenience. Let $G = (V, E)$ be an arbitrary connected graph with vertex set $V = \{1, \dots, n\}$. For a given subset $V_0 \subset V$ of vertices, we define $\langle V_0 \rangle \subset G$ to be the subgraph induced by V_0 . For a function $f \in l^2(G)$, let $\text{supp}(f) := \{i \in V \mid f(i) \neq 0\}$ and $\text{supp}_\pm(f) = \{i \in \text{supp}(f) \mid \pm f(i) > 0\}$. We say that a non-zero function f in a subspace $\mathfrak{E} \subset l^2(G)$ has *minimal support*, if for every non-zero function $g \in \mathfrak{E}$ with $\text{supp}(g) \subset \text{supp}(f)$ we have $\text{supp}(g) = \text{supp}(f)$.

Let $\mathcal{M}(G)$ be the set of all symmetric (not necessarily stochastic) matrices $M = (m_{ij})$ with all non-diagonal entries $m_{ij} > 0$ if $i \sim j$ and $m_{ij} = 0$ if $i \not\sim j$. Note that we do not impose any sign conditions on the diagonal entries m_{ii} . It is a direct consequence of the connectedness of G and Perron-Frobenius that the highest eigenvalue $\lambda_0(M)$ is simple. Colin de Verdière calls the matrices in $\mathcal{M}(G)$ *Schrödinger operators on the graph G* (see [8]), and they play an important role for his graph invariant $\mu(G)$ (see [7]). The following result can be considered as a graph theoretical analogue of the Courant nodal domain for Riemannian manifolds (see [6]):

Proposition 4.1 ([17]). *Let $G = (V, E)$ be a finite connected graph and $M \in \mathcal{M}(G)$. Let $\mathfrak{E} = \mathfrak{E}_{\lambda_1(M)}$ be the eigenspace of the second highest eigenvalue of M . Let $f \in \mathfrak{E}$ be a function of minimal support. Then $\langle \text{supp}_+(f) \rangle$ and $\langle \text{supp}_-(f) \rangle$ are both connected graphs.*

This fact allows us to prove the following special result:

Corollary 4.2. *Let $\Gamma \in \{A_3, B_3, H_3\}$ and G be the associated Cayley graph with respect to the canonical set $S = \{s_1, s_2, s_3\}$ of generators. Let X be an interior point of Δ_Γ . Then we have $\lambda_1 = \lambda_1(X) \in [0, 1)$, and the corresponding eigenspace has dimension ≤ 3 .*

Proof (following mainly [17]): Let \mathfrak{E} be the eigenspace of λ_1 . Since $X \in \text{int}(\Delta)$, we have $P_X \in \mathcal{M}(G)$. Note that the spectrum of P_X is symmetric (since G is bipartite), and therefore we must have $\lambda_1 \in [0, 1)$, since both eigenvalues $-1, 1$ are simple, because $G_{P_X} = G$ is connected.

Recall that $V = \Gamma$ and that $G = (V, E)$ is the one-skeleton of one of the solids $(4, 6, 6)$, $(4, 6, 8)$ or $(4, 6, 10)$. In particular, G is a 3-connected finite planar graph of constant vertex degree three. We think of the elements of G as being enumerated and identify group elements with their corresponding integers. Thus, it makes sense to write $p_{\gamma, \gamma'}$ for the matrix entries of P_X .

Let $\gamma_0 \in V$ be arbitrary and $\Theta_{\gamma_0} : \mathfrak{E} \rightarrow \mathbb{R}^3$ be the map

$$(20) \quad \Theta_{\gamma_0}(f) = (f(\gamma_0 s_1), f(\gamma_0 s_2), f(\gamma_0 s_3)).$$

We prove that this map is *injective*, which shows that $\dim \mathfrak{E} \leq 3$. Assume that there is a non-zero $f \in \mathfrak{E}$ with $\Theta(f) = 0$, i.e., $\text{supp}(f) \cap \gamma_0 S = \emptyset$. Choose a function $g \in \mathfrak{E}$ with minimal support $\text{supp}(g) \subset \text{supp}(f)$.

We first show that $g(\gamma_0) = 0$. Assume that $g(\gamma_0) \neq 0$. Without loss of generality, we can assume that $\gamma_0 \in \text{supp}_+(g)$ (otherwise replace g by $-g$).

Since

$$\lambda_1 g(\gamma_0) = \sum_{j=1}^3 p_{\gamma_0, \gamma_0 s_j} g(\gamma_0 s_j) = 0,$$

we must have $\lambda_1 = 0$. Since $\text{supp}_+(g)$ is connected by Proposition 4.1 and g vanishes on all neighbours of γ_0 , we conclude that $\text{supp}_+(g) = \{\gamma_0\}$. Let $S_n(\gamma) \subset V$ denote the sphere of combinatorial radius n around γ . Since for our graphs, all vertices in $S_1(\gamma_0)$ have two neighbours in $S_2(\gamma_0)$ and g is an eigenfunction to the eigenvalue zero, we must have $f(\gamma') \leq 0$ for all $\gamma' \in S_2(\gamma_0)$, and there exists a $\gamma_1 \in S_2(\gamma_0)$ with $f(\gamma_1) < 0$. Now, γ_1 cannot be a neighbour of all three vertices in $S_1(\gamma_0)$, and therefore must have a neighbour γ_2 with distance at least 2 to γ_0 . Again, since g is an eigenfunction to the eigenvalue zero, γ_2 must have a neighbour γ_3 with $g(\gamma_3) > 0$. Therefore, $\gamma_3 \in \text{supp}_+(g) \setminus \{\gamma_0\}$, which is a contradiction.

So we proved $g(\gamma_0) = 0$. Let $\gamma' \in \text{supp}(g)$. Since G is 3-connected, there are three pairwise disjoint paths P_1, P_2, P_3 , connecting γ_0 with γ' . Without loss of generality, we can assume that the path P_i contains the vertex $\gamma_0 s_i$. Starting in $\gamma_0 s_i$ and following the path P_i in direction γ' , let $\gamma_i \in P_i$ be the first vertex with $g(\gamma_i) = 0$ and γ_i adjacent to $\text{supp}(g)$. Since g is an eigenfunction, γ_i must be adjacent to both $\text{supp}_+(g)$ and $\text{supp}_-(g)$. Now, contract $\text{supp}_+(g)$ and $\text{supp}_-(g)$ to single vertices, denoted by v_+ and v_- (which is possible since both sets are connected, by Proposition 4.1) and contract also the parts of the paths P_1, P_2, P_3 from γ_0 to γ_i , and remove all other vertices on which g vanishes. The resulting graph is planar and contains $K_{3,3}$ as a subgraph (where one set of vertices are $\gamma_1, \gamma_2, \gamma_3$ and the other set are γ_0, v_+, v_-), which is a contradiction. \square

Now we recall a general method, described in [9, Chapter 3E.2], to calculate the spectrum of special doubly stochastic matrices: Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a finite group and $p : \Gamma \rightarrow [0, 1]$ be a probability vector on Γ , i.e., $\sum_{i=1}^n p(\gamma_i) = 1$. Let $P = (p_{ij})$ be the doubly stochastic matrix defined by $p_{ij} = p(\gamma_j^{-1} \gamma_i)$. The Fourier transform $\widehat{p}(\rho)$ of p at a given irreducible representation $\rho : \Gamma \rightarrow \text{GL}(d_\rho, \mathbb{C})$ is defined as

$$(21) \quad \widehat{p}(\rho) = \sum_{i=1}^n p(\gamma_i) \rho(\gamma_i).$$

Let $f_\rho(t)$ be the characteristic polynomial of $\widehat{p}(\rho)$. Then the spectrum $\sigma(P)$ is the union of the zeros of the polynomials $f_\rho(t)$, where $\rho : \Gamma \rightarrow \text{GL}(d_\rho, \mathbb{C})$ runs through all non-equivalent irreducible representations. Each zero of $f_\rho(t)$ appears in $\sigma(P)$ with multiplicity d_ρ .

We apply this to our matrix P_X in the case $\Gamma \in \{A_3, B_3, H_3\}$. For $X = (x, y, z) \in \text{int}(\Delta)$, we choose the probability vector

$$p_X(\gamma) = \begin{cases} x, & \text{if } \gamma = s_1, \\ y, & \text{if } \gamma = s_2, \\ z, & \text{if } \gamma = s_3, \\ 0, & \text{otherwise,} \end{cases}$$

and conclude that $P = P_X$. Let $\rho : \Gamma \rightarrow O(3)$ be the geometric realisation, and $f_X(t)$ be the characteristic polynomial of $\hat{p}(\rho) = x\sigma_1 + y\sigma_2 + z\sigma_3 \in \mathbb{R}^{3 \times 3}$. $f_X(t)$ is a polynomial in t whose coefficients are polynomials in x, y, z . The zeros of f_X are continuous functions in $X = (x, y, z)$, which we denote by $\mu_1(X) \geq \mu_2(X) \geq \mu_3(X)$, to distinguish them from the eigenfunctions $\lambda_i(X)$. Note that each $\mu_i(X)$ is an eigenvalue of P_X of multiplicity at least three. Let us now focus on the case $\Gamma = H_3$ (the Coxeter groups A_3 and B_3 are treated analogously). A straightforward calculation yields

$$f_X(t) = t^3 - t^2 - q(x, y, z)t + q(x, y, z) + 2(2 - \varphi)xyz,$$

where $q(x, y, z) = 1 - 4xy - 3xz - (3 - \varphi)yz$. For $\hat{X} = (1/3, 1/3, 1/3)$ we obtain

$$f_{\hat{X}}(t) = \left(t - \frac{1}{3}\right) \left(t^2 - \frac{2}{3}t - \frac{\varphi + 1}{9}\right).$$

The results in [16] for the canonical Laplacian show that $3\lambda_1(\hat{X})$ has multiplicity three and is the largest zero of the polynomial $t^2 - 2t - (\varphi + 1)$. This shows that $\lambda_1(\hat{X}) = \mu_1(\hat{X})$. Since the multiplicity of λ_1 is bounded from above by three and the multiplicity of μ_1 is bounded from below by three and both functions are continuous, they must coincide and represent an eigenvalue of P_X of multiplicity three. It remains to show

$$(22) \quad \lambda_1(\hat{X}) = \Psi_\lambda \circ \Psi_\Delta^{-1}(\hat{X}),$$

which implies, again by the continuity of both functions, that $\lambda_1 = \Psi_\lambda \circ \Psi_\Delta^{-1}$. The validity of (22) is shown at the beginning of the next section, finishing the proof of the conjecture for the Coxeter groups A_3, B_3 and H_3 . \square

Remark 4.3. Let $X \in \text{int}(\Delta)$, and \mathfrak{E} be the eigenspace of P_X to the eigenvalue $\lambda_1(X)$. The above arguments show that, for all $\gamma \in \Gamma$, the maps $\Theta_\gamma : \mathfrak{E} \rightarrow \mathbb{R}^3$ (given by (20)) are bijective. This fact is equivalent to a particular transversality property of P_X , the so-called Strong Arnold Hypothesis (for the precise definition see, e.g., [7] or [12]). The Strong Arnold Hypothesis played a crucial role in the proof that Colin de Verdière's graph invariant is monotone with respect to taking minors.

5. PROOFS OF THE RESULTS ABOUT THE ARCHIMEDEAN SOLIDS

Our first goal is the proof of the identity (22). We specify the calculations in Section 3 to the case $\Gamma \in \{A_3, B_3, H_3\}$. Some explicit constants are collected in the following table:

Γ	A_3	B_3	H_3
η	1	$\sqrt{2}$	$\varphi = \frac{1+\sqrt{5}}{2}$
$\rho = 3 - \eta^2$	2	1	$2 - \varphi$
Λ	$2 + \sqrt{2}$	$4 + 2\sqrt{3}$	$\frac{2}{2 - \sqrt{2} + \varphi}$
λ	$\frac{1+\sqrt{2}}{3}$	$\frac{1+\sqrt{3}}{3}$	$\frac{1+\sqrt{2}+\varphi}{3}$

Let ρ and η be given as in the table. Then we have

$$(23) \quad M = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{\eta}{2} \\ -\frac{1}{2} & -\frac{\eta}{2} & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{\rho} \begin{pmatrix} 1 + \rho & \eta & 2 \\ \eta & 3 & 2\eta \\ 2 & 2\eta & 4 \end{pmatrix}$$

and $V^2 = \frac{\rho}{4}$.

Specialisation of (19) to the case $\hat{X} = (1/3, 1/3, 1/3)$ yields

$$\frac{1}{3\mu V} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = M^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Let Λ denote the Perron-Frobenius eigenvalue of the positive matrix M^{-1} (see the table for the explicit value). Using $\lambda = x + y + z - 2V\mu = 1 - \frac{2}{3\Lambda}$, we observe that $\lambda = \Psi_\lambda \circ \Psi_\Delta^{-1}(\hat{X}) = \frac{1+\sqrt{4-\rho}}{3}$ in all three cases. Consulting [16], we see that $3\lambda_1(\hat{X})$ is the largest zero of the polynomial

$$f(t) = \begin{cases} t^2 - 2t - 1, & \text{if } \Gamma = A_3, \\ t^2 - 2t - 2, & \text{if } \Gamma = B_3, \\ t^2 - 2t - (\varphi + 1), & \text{if } \Gamma = H_3. \end{cases}$$

Since 3λ is the largest zero of $f(t)$ in all three cases, we conclude that

$$\lambda_1(\hat{X}) = \Psi_\lambda \circ \Psi_\Delta^{-1}(\hat{X}).$$

□

Proof of Theorem 1.10: Let $\Gamma \in \{A_3, B_3, H_3\}$ and $G = (V, E)$ be the Cayley graph associated to Γ with respect to the canonical set $S = \{s_1, s_2, s_3\}$ of generators. Recall that G is the one-skeleton of the Archimedean solids $(4, 6, 6)$, $(4, 6, 8)$ and $(4, 6, 10)$, respectively.

Let $p = \alpha p_1 + \beta p_2 + \gamma p_3$ be a general point in the spherical fundamental domain $\mathcal{F}_0 \subset S^2$. Choosing $\mu = 1$ and using (16) and (23), we obtain

$$(24) \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \frac{1}{2\sqrt{\rho}} \begin{pmatrix} 1 + \rho + \frac{\eta\beta + 2\gamma}{\alpha} \\ 3 + \frac{\alpha + 2\gamma}{\beta}\eta \\ 4 + \frac{2\alpha + 2\eta\beta}{\gamma} \end{pmatrix}$$

and $\lambda' = x' + y' + z' - \sqrt{\rho}$. $\Psi_\Delta(p)$ and $\Psi_\lambda(p)$ are then given by the expressions in (18).

Since the lengths of the Euclidean edges are given by $\|p - \sigma_1(p)\| = 2\alpha V$, $\|p - \sigma_2(p)\| = 2\beta V$ and $\|p - \sigma_3(p)\| = 2\gamma V$ (see Corollary 1.6), there is only one point $p_0 \in \mathcal{F}_0$ for which all edges are of equal length, namely the choice $\alpha = \beta = \gamma$. Using (24) in this case and calculating $(x, y, z) = \Psi_\Delta(p_0)$ and $\lambda = \Psi_\lambda(p_0)$ with the help of (18) yields

$$(25) \quad (x, y, z) = \frac{1}{12 + \rho + 6\eta}(3 + \rho + \eta, 3 + 3\eta, 6 + 2\eta) \quad \text{and} \quad \lambda = \frac{12 + 6\eta - \rho}{12 + 6\eta + \rho}.$$

By Theorem 1.3 and Proposition 1.8, this is the only critical point of $\lambda_1 : \text{int}(\Delta) \rightarrow (0, 1)$. By Proposition 1.2, λ_1 has a global minimum in $\text{int}(\Delta)$, which must therefore agree with (25).

From Corollary 1.6 and Proposition 1.8 we conclude that $\lambda_1 : \text{int}(\Delta) \rightarrow (0, 1)$ is analytic, and we know from the proof of Proposition 1.2 that λ_1 is convex. Assume that λ_1 would not be *strictly convex*. Then there would exist three different collinear points $X_1, X_2, X_3 \in \text{int}(\Delta)$ with $\lambda_1(X_1) = \lambda_1(X_2) = \lambda_1(X_3)$. Convexity of λ_1 would force λ_1 to be constant on the line segment bounded by the two extremal points of X_1, X_2, X_3 . Analyticity of λ_1 would imply that λ_1 is constant along the whole line in Δ containing these three points. But this would lead to $\lambda_1(X_1) = \lambda_1(X_2) = \lambda_1(X_3) = 1$, a contradiction to $\lambda_1 < 1$ on the interior of Δ .

In the case $\Gamma = H_3$, i.e., $(\eta, \rho) = (\varphi, 2 - \varphi)$, we obtain

$$X_0 = (x, y, z) = \frac{1}{14 + 5\varphi}(5, 3 + 3\varphi, 6 + 2\varphi) \quad \text{and} \quad \lambda = \frac{10 + 7\varphi}{14 + 5\varphi}.$$

The corresponding spectral representation Φ_{X_0} agrees, up to the factor $\frac{|\Gamma|}{3}$, with the orbit map $\Phi(\gamma) = \gamma p_0$, by Corollary 1.6, and is therefore faithful.

Analogously, one easily checks that the choices $(\eta, \rho) = (1, 2)$ and $(\eta, \rho) = (\sqrt{2}, 1)$ lead to the explicit values for (x, y, z) and λ , given in Remark 1.11. \square

Proof of Theorem 1.12: We only discuss the Archimedean solid $(4, 6, 10)$ (i.e., $\Gamma = H_3$), the other solids are treated analogously.

Note, by the construction of p_1, p_2, p_3 in (7), that the orbit Γp_1 gives the vertices of an icosahedron. Up to a scalar factor, p_2 points to the centre of a face of this icosahedron and p_3 to the midpoint of an edge of the icosahedron, and the orbits Γp_2 and Γp_3 are the vertices of an dodecahedron $(5, 5, 5)$ and of an icosidodecahedron $(3, 5, 3, 5)$, respectively. Moreover, it is easy to see that there are positive constants $0 < c_0 < C_0$ such that $\alpha p_1 + \beta p_2 + \gamma p_3 \in \mathcal{F}_0$, $\alpha, \beta, \gamma \geq 0$, implies $c_0 \leq \alpha + \beta + \gamma \leq C_0$.

Let $X_n = (x_n, y_n, z_n) \in \text{int}(\Delta)$ be a sequence converging to $(x, 0, z) \in \Delta$ with $x, z > 0$. Then there are constants $c_1, c_2 > 0$ with $c_1 x_n < z_n < c_2 x_n$. Let $\Psi_\Delta^{-1}(X_n) = q_n = \alpha_n p_1 + \beta_n p_2 + \gamma_n p_3 \in \mathcal{F}_0$. Our aim is to show $\alpha_n, \gamma_n \rightarrow$

0. From (24) and (18), we deduce that

$$\begin{aligned} x_n &= \frac{1}{F(\alpha_n, \beta_n, \gamma_n)} \left(3 - \varphi + \frac{\varphi\beta_n + 2\gamma_n}{\alpha_n} \right), \\ y_n &= \frac{1}{\beta_n F(\alpha_n, \beta_n, \gamma_n)} (3\beta_n + (\alpha_n + 2\gamma_n)\varphi), \\ z_n &= \frac{1}{F(\alpha_n, \beta_n, \gamma_n)} \left(4 + \frac{2\alpha_n + 2\varphi\beta_n}{\gamma_n} \right), \end{aligned}$$

with

$$F(\alpha, \beta, \gamma) = 10 - \varphi + \frac{\varphi\beta + 2\gamma}{\alpha} + \frac{\alpha + 2\gamma}{\beta}\varphi + \frac{2\alpha + 2\varphi\beta}{\gamma}.$$

Since $y_n \rightarrow 0$ and $c_0 \leq 3\beta_n + (\alpha_n + 2\gamma_n)\varphi \leq 4C_0$, we must have $\beta_n F(\alpha_n, \beta_n, \gamma_n) \rightarrow \infty$. This necessarily implies $\alpha_n \gamma_n \rightarrow 0$. Assume α_n converges to zero on a subsequence, on which γ_n does not converge to zero. Then $F(\alpha_n, \beta_n, \gamma_n) \rightarrow \infty$ implies that z_n converges to zero on a finer subsequence and, since $c_1 x_n < z_n$, x_n must also converge to zero on this finer subsequence, contradicting to $x_n + y_n + z_n = 1$. This shows that both $\alpha_n, \gamma_n \rightarrow 0$, i.e., q_n converges to a multiple of p_2 . By Corollary 1.6, the corresponding spectral representations converge, up to a scalar factor, to the orbit map $\Phi(\gamma) = \gamma p_2$, and Γp_2 are the vertices of a dodecahedron. This proves the convergence behaviour as X_n converges to an interior point of the bottom edge of the simplex Δ in Figure 2. The converge behaviour to interior points of the other two edges of Δ is proved analogously.

The curve C_2 is characterised by the property $\alpha = \gamma$. Using this fact and the relation (24), and substituting $t = \frac{\alpha}{\beta}$ we obtain

$$C_2 = \left\{ \frac{1}{3\varphi t^2 + (14 - \varphi)t + 3\varphi} \begin{pmatrix} (5 - \varphi)t + \varphi \\ 3\varphi t^2 + 3t \\ 6t + 2\varphi \end{pmatrix} \mid t \in (0, \infty) \right\} \subset \Delta.$$

Note that $t \rightarrow \infty$ implies $\beta \rightarrow 0$, which means that the Euclidean edges between the 4-gons and the 10-gons shrink to zero and the corresponding spectral representations converge to equilateral realisations of the buckyball (5, 6, 6) (see Figure 3). The convergence behaviour along the other curves C_1, C_3 is proved analogously. \square

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